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ASYMPTOTICS OF MULTINOMIAL SUMS AND IDENTITIES BETWEEN MULTI-INTEGRALS

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ABSTRACT

We calculate the asymptotics of combinatorial sums $\sum_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta}$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_h)$ with $\alpha_i = \alpha_j$ for certain i, j. Here h is fixed and the α_i's are natural numbers. This implies the asymptotics of the corresponding S_n-character degrees $\sum_{\lambda} f(\lambda)d_{\lambda}^{\beta}$. For certain sequences of S_n characters which involve Young's rule, the latter asymptotics were obtained earlier [1] by a different method. Equating the two asymptotics, we obtain equations between multi-integrals which involve Gaussian measures. Special cases here give certain extensions of the Mehta integral [5], [6].

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Introduction

The present work extends the asymptotics obtained in [3] and derives applications for the evaluation of certain multi-integrals with Gaussian measure. In particular, extensions of certain special cases of the Mehta integral are derived here.

The Mehta integral, [5], [6], which is a consequence of the celebrated Selberg integral [4], [5], [6], [9], states that

$$\int \cdots \int_{\mathbb{R}^k} \left[\left(\prod_{1\le i<j\le k} |x_i - x_j| \right) \cdot \exp\left(-\frac{1}{2} \sum_{i=1}^{k} x_i^2 \right) \right]^{\beta} d^{(k)}x$$

$$= \left(\sqrt{2\pi}\right)^k \cdot \beta^{-\frac{k}{2} - \frac{\beta k(k-1)}{4}} \cdot [\Gamma(1+\frac{1}{2}\beta)]^{-k} \cdot \prod_{j=1}^{k} \Gamma(1+\frac{1}{2}\beta j).$$

Here Γ is the Gamma function.

Let $\Omega_k = \{(x_1,\ldots,x_k) \in \mathbb{R}^k \mid x_1+\cdots+x_k = 0 \text{ and } x_1 \ge \cdots \ge x_k\}$. In section 4 of [7] we saw how to change the domain of integration in the Mehta integral from Ω_k to $\mathbb{R}^k$; the domain is already $\mathbb{R}^k$ in the form of the integral just stated. Thus, Theorem 3.3 below, which relates the Mehta integral I' to the multi-integral I there, extends a special case of the Mehta integral. Note that Theorem 3.3 is a special case of Theorem 3.2 here, which also relates two such multi-integrals.

The evaluations and the equations between these multi-integrals are by-products of the study of the asymptotics of the degrees of certain S_n-character sequences, which we now describe.

Also in [3] we obtained the asymptotics of

$$\sum_{\boldsymbol{\alpha}\in A_h(n)} f(\boldsymbol{\alpha}) \binom{n}{\boldsymbol{\alpha}}^{\beta}$$

as $n \to \infty$ and h fixed. Here

$$A_h(n) = \{(\alpha_1,\ldots,\alpha_h) \mid 0 \le \alpha_i \in \mathbb{Z} \text{ and } \sum \alpha_i = n\}.$$

In [3] we gave several applications to the evaluation of certain multi-integrals.

Similar sums, but with $\alpha = (\alpha_1,\ldots,\alpha_h)$ having $\alpha_i = \alpha_j$ for certain i,j, arise naturally. In the present paper we consider the asymptotics of such sums and give some applications.

More specifically, let $r_1,\ldots,r_p$ be positive integers with $r_1+\cdots+r_p = h$, let $r_0 = 0$ and

$$\theta_i = \{r_1+\cdots+r_{i-1}+1,\ldots,r_1+\cdots+r_i\}, \text{ so } \{1,\ldots,h\} = \bigcup_{i=1} \theta_i.$$

Denote $s \sim t$ if $s, t \in \theta_i$ for some i, and

$$B_\theta(n) = \{\boldsymbol{\alpha} \in A_h(n) \mid \alpha_s = \alpha_t \text{ if } s \sim t\}.$$

Theorem 1.2 gives the asymptotics of the sums

$$\sum_{\boldsymbol{\alpha} \in B_\theta(n)} f(\boldsymbol{\alpha}) \binom{n}{\boldsymbol{\alpha}}^\beta \qquad (n \to \infty, \quad \theta \text{ fixed}).$$

Here we restrict ouselves to functions $f(\boldsymbol{\alpha})$ which are products of terms of the form $(\alpha_i - \alpha_j + d_{ij})$ and $(\alpha_i + d_i)$, where d_{ij} and d_i are constants, and $i, j = 1, \ldots, h$. Note that in [3] we considered a more general class of functions $f(\boldsymbol{\alpha})$, but the functions considered in the present paper suffice for the applications. Moreover, restricting ourselves to these functions considerably simplifies the discussion of [3], namely, it allows one to avoid introducing permissible functions in the sense of [3].

Let now $\mathrm{Par}(n)$ denote the partitions of n and $\Lambda_\theta(n) = B_\theta(n) \cap \mathrm{Par}(n)$. As an application we obtain, in Theorem 2.1 below, the asymptotics of

$$\sum_{\lambda \in \Lambda_\theta(n)} f(\lambda) d_\lambda^\beta$$

(where d_λ equals the number of standard Young tableaux of shape λ).

This is applied to study the asymptotics of $\deg(y^\ell(\eta^q)_n)$, an object we now describe.

Let S_n denote the nth symmetric group, and for each n let ψ_n be an S_n-character. Sequences $\psi = \{\psi_n\}_{n \geq 0}$ arise naturally in Representation Theory. A useful tool for studying such sequences is the notion of "Young derived sequences", introduced in [8]: For each $\lambda \in \mathrm{Par}(n)$, χ_λ is the corresponding irreducible S_n-character (so $\chi_{(n)}$ is the trivial S_n-character). Given $\psi = \{\psi_n\}_{n \geq 0}$ as above, its "Young derived sequence" $y(\psi)$ is defined via $y_n(\psi) = (y(\psi))_n = \sum_{j=0}^{n} \psi_j \hat{\otimes} \chi_{(n-j)}$, where $\hat{\otimes}$ is the "outer" product of characters. Also, $y^\ell(\psi)$ is the ℓth derived such sequence. For example, let $\psi_0 = 1$, $\psi_n = 0$ if $n \geq 1$. Also let $\dim V = \ell$ and let $\varphi_n^{(\ell)}$ denote the S_n-character given by the classical action of S_n on $V^{\otimes n}$. Then $(y^\ell(\psi))_n = \varphi_n^{(\ell)}$ (Example 1.4 of [8]).

Let

$$\eta = \{\eta_n\}, \quad \eta_n = \sum_{\lambda \in \Lambda_k(n)} b(\lambda) \chi_\lambda$$

($\Lambda_k(n)$ are the partitions of n with at most k parts) and denote

$$\eta^q = \{\eta_n^q\}, \quad \eta_n^q = \sum_{\lambda \in \Lambda_k(n)} b(\lambda) \chi_{\lambda^q},$$

where

$$\lambda = (\lambda_1, \lambda_2, \ldots) \quad \text{and} \quad \lambda^q = (\underbrace{\lambda_1, \ldots, \lambda_1}_{q}, \underbrace{\lambda_2, \ldots, \lambda_2}_{q}, \ldots).$$

The sequences $y^\ell(\eta^q)$ are studied in [1] and [2]. The asymptotics of $\deg(y^\ell(\eta^q))_n$ are given by [1] Theorem 3.3, while the relations between the coefficients in η and in $y^\ell(\eta^q)$, $1 \leq \ell \leq q-1$, are given by Theorem 1.2 of [2] (which generalizes Example 1.4 of [8]).

Theorem 2.1 below together with Theorem 1.2 of [2] allow us to compute the asymptotics of $\deg(y^\ell(\eta^q))_n$ $(n \to \infty)$ in a way which is independent of [1, Theorem 3.3]. These two computations lead to $\deg(y^\ell(\eta^q))_n \simeq c_1 I_1 n^u (qk+\ell)^n$ [2, Theorem 4.1] and $\deg(y^\ell(\eta^q))_n \simeq c_2 I_2 n^u (qk+\ell)^n$ (Proposition 3.1 below). Here u is a certain number, c_1, c_2 are explicit constants, and I_1, I_2 are multi-integrals involving Vandermonde-like polynomials and Gaussian measures.

Equating the two asymptotics we deduce identities of the form

$$I_1 = (c_2/c_1) I_2 \quad \text{(see Theorem 4.3 below).}$$

Note that the results of [3] sufficed for that second asymptotic computation with the resulting integral identity only for the case $\ell = q-1$ [2, Theorems 4.3, 4.4]. However, Theorem 2.1 below allows us to deduce corresponding calculations and multi-integral identities for all $1 \leq \ell \leq q-1$ (Theorems 3.2, 3.3 below).

Certain choices of $f(\lambda)$ give I_1 as the Mehta integral (Theorem 3.3 here), thus enabling the evaluation of I_2, which to our knowledge is a new result, and a variant of (a special case of) the Mehta–Selberg integral.

In §4 we prove that certain homogeneous polynomials of the differences $x_i - x_j$ do satisfy a property ("niceness" in the sense of [8]) which then allows us to obtain both corresponding asymptotics of $\deg(y^\ell(\eta^q))_n$ also when $q \leq \ell$. Hence, in Theorem 4.3 below, we are able to deduce further equations between corresponding multi-integrals which involve Gaussian measures.

As mentioned above, the S_n-characters $(y^\ell(\eta^q))_n$ generalize the classical S_n-character $(y^\ell(\psi))_n$ of [8, Expl. 1.4]. The multiplicity of χ_λ in $(y^\ell(\psi))_n$ is $s_\ell(\lambda)$, the number of ℓ-semi-standard tableaux of shape λ. Theorem 1.2 of [2] gives the multiplicities $b^{(\ell)}(\mu)$ of χ_μ in $(y^\ell(\eta^q))_n$, but only when $\ell \leq q-1$. When $q-1 \leq \ell$, Theorem 4.5 below gives an approximation of $b^{(\ell)}(\mu)$ by a polynomial $a^{(\ell-\mu+1)}(\mu)$, where $a^{(s)}(x)$ is obtained from an explicit polynomial $a^{(0)}(x)$ by "partition"-integrating $a^{(0)}(x)$ s times.

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1. Asymptotics for multinomial sums

In this section we calculate the asymptotics as $n \to \infty$, of $\sum_{\underline{\alpha} \in B_\theta(n)} f(\underline{\alpha}) \binom{n}{\boldsymbol{\alpha}}^\beta$, when f is essentially a monomial in the $\alpha_s - \alpha_t$'s and in the α_t's and $B_\theta(n)$ are the $(\alpha_1, \ldots, \alpha_n)$'s, $\alpha_1 + \cdots + \alpha_s = n$, with some "$\theta$" identifications. This is Theorem 1.2 below, which "θ" generalizes [3, Thm. 1] for such f's. In comparison to [7] and to [3], the restriction to such f's considerably simplifies the calculations while all the applications known to us so far involve only such f's. It is quite clear that with some more work, Theorem 1.2 can be proved for a much wider class of functions f.

We define the following:

Notations 1.1: $\mathbb{N} = \{0, 1, 2, \ldots\}$,

$$A_h(n) = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_h) | \forall \ \alpha_i \in \mathbb{N} \text{ and } \alpha_1 + \cdots + \alpha_h = n\}.$$

Let $r_1, \ldots, r_p \in \mathbb{N} - \{0\}$, $r_1 + \cdots + r_p = h$, $r_0 = 0$, and

$$\theta_i = \{r_1 + \cdots + r_{i-1} + 1, \ldots, r_1 + \cdots + r_i\},$$

so that $\{1, \ldots, h\} = \bigcup_{i=1}^p \theta_i$. Denote

$$B_\theta(n) = \{\underline{\alpha} \in A_n(n) | \ \alpha_s = \alpha_t \text{ if } \exists 1 \le i \le p \text{ with } s, t \in \theta_i\}.$$

Define $s \underset{\theta}{\sim} t$ if there exists $1 \le i \le p$ with $s, t \in \theta_i$; otherwise $s \underset{\theta}{\nsim} t$. Thus

$$B_\theta(n) = \{\alpha \in A_n(n) | \ \alpha_s = \alpha_t \text{ if } s \underset{\theta}{\sim} t\}.$$

In the sequel we denote $\underset{\theta}{\sim}$ by $\sim$ and $\underset{\theta}{\nsim}$ by $\nsim$. Let $a_t, a_{st} \in \mathbb{N}$, $1 \le s, t \le h$, and fix

$$f(\underline{\alpha}) = \left[\prod_{1 \le s,t \le h} \prod_{v=1}^{a_{st}} (\alpha_s - \alpha_t + d_{stv}) \right] \cdot \left[\prod_{t=1}^{h} \prod_{v=1}^{a_t} (\alpha_t + d_{tv})^{\varepsilon_{tv}} \right],$$

where the d_{st}'s and the d_t's are constants and $\varepsilon_{tv} \in \{0, \pm 1\}$. Clearly, if $\boldsymbol{\alpha} \in B_\theta(n)$ then $f(\boldsymbol{\alpha}) = d \cdot f_0(\boldsymbol{\alpha})$, where $d = \prod_{s \sim t} \prod_{v=1}^{a_{st}} d_{stv}$ and

$$f_0(\boldsymbol{\alpha}) = \left[\prod_{s \nsim t} \prod_{v=1}^{a_{st}} (\alpha_s - \alpha_t + d_{stv}) \right] \cdot \left[\prod_{t=1}^{h} \prod_{v=1}^{a_t} (\alpha_t + d_t)^{\varepsilon_{tv}} \right].$$

Denote $\sum_{v=1}^{a_t} \varepsilon_{tv} = b_t$. Finally, recall that $w_n \underset{n \to \infty}{\simeq} z_n$ if $\lim_{n \to \infty}(w_n / z_n) = 1$. We can now state

THEOREM 1.2: *Recall that h is the length of $\underline{\alpha}$, and that the a_t's and the a_{st}'s determine the factors of the monomial $f(\underline{\alpha})$. As $n \to \infty$,*

$$\sum_{\boldsymbol{\alpha}\in B_\theta(n)} f(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta} \simeq c\cdot I\cdot n^u\cdot h^{\beta n},$$

where

$$u = -\frac{\beta h}{2} + \frac{1}{2}\sum_{1\le s\nsim t\le h} a_{st} + \sum_{t=1}^{h} b_t + \frac{1}{2}(p-1) + \frac{\beta}{2},$$

$$c = \left(\frac{1}{\sqrt{2\pi}}\right)^{\beta(h-1)} \cdot h^{-u+\frac{\beta}{2}} \cdot \prod_{1\le s\sim t\le h}\ \prod_{v=1}^{a_{st}} d_{stv},$$

and

$$I = \int_{r_1x_1+\cdots+r_px_p=0}\!\!\cdots\cdots\!\int \prod_{1\le i\ne j\le p}(x_i-x_j)^{e_{ij}}\cdot \exp\left(-\frac{\beta}{2}\sum_{i=1}^{p} r_i x_i^2\right) d^{(p-1)}x.$$

Here

$$e_{ij} = \sum_{s\in\theta_i, t\in\theta_j} a_{st}.$$

Proof: The proof consists of the following four steps:

STEP 1: Write $f(\boldsymbol{\alpha}) = d\cdot f_0(\boldsymbol{\alpha})$ as above, $f_0(\boldsymbol{\alpha}) = P_1\cdot P_2$, where

$$P_1 = \prod_{1\le s\nsim t\le h}\ \prod_{v=1}^{a_{st}}(\alpha_s-\alpha_t+d_{stv}) \quad\text{and}\quad P_2 = \prod_{t=1}^{h}\ \prod_{v=1}^{a_t}(\alpha_t+d_{tv})^{\varepsilon_{tv}}.$$

Expand P_1: $P_1 = \prod_{1\le s\nsim t\le h}(\alpha_s-\alpha_t)^{a_{st}} + P_1^*$, where P_1^* involves the other terms of P_1; those are clearly of lower degree in the $\alpha_s - \alpha_t$'s. Finally, write $f_0(\boldsymbol{\alpha}) = f_1(\boldsymbol{\alpha}) + f_2(\boldsymbol{\alpha})$, where $f_1(\boldsymbol{\alpha}) = \prod_{1\le s\nsim t\le h}(\alpha_s-\alpha_t)^{a_{st}}\cdot P_2$ and $f_2(\boldsymbol{\alpha}) = P_1^*\cdot P_2$.

We shall prove Theorem 1.2 with $f_1(\boldsymbol{\alpha})$ replacing $f_0(\alpha)$. In that proof, notice how each term $\alpha_s - \alpha_t$ in $f_1(\boldsymbol{\alpha})$ contributes a $\sqrt{n}$ to the asymptotics. Hence, expanding $f_2(\boldsymbol{\alpha})$ and computing the corresponding asymptotics, one obtains the same exponential growth $h^{\beta n}$, but a smaller power of n, namely $n^{u'}$, where $u' \nleq u$. It follows that

$$\sum_{\boldsymbol{\alpha}\in B_\theta(n)} f_0(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta} \underset{n\to\infty}{\simeq} \sum_{\boldsymbol{\alpha}\in B_\theta(n)} f_1(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta}.$$

STEP 2: Let $\boldsymbol{\alpha} \in A_h(n)$ and define $c(\boldsymbol{\alpha}) = (c_1(\boldsymbol{\alpha}), \ldots, c_h(\boldsymbol{\alpha}))$ via $\alpha_t = n/h + c_t(\boldsymbol{\alpha})\sqrt{n}$. Given $0 < \rho \in \mathbb{R}$, denote

$$B_\theta(n,\rho) = \{\boldsymbol{\alpha} \in B_\theta(n) \bigm| |c_t(\boldsymbol{\alpha})| < \rho,\ t = 1, \ldots, h\}.$$

For fixed ρ and for n large, $\alpha_t \simeq n/h$, $t = 1, \ldots, h$, so α_t is large, hence Stirling's formula applies to $\alpha_t!$. Moreover, for such $\boldsymbol{\alpha}$,

$$f_1(\boldsymbol{\alpha}) \simeq \left[\prod_{1 \le s \nsim t \le h} (c_s(\boldsymbol{\alpha}) - c_t(\boldsymbol{\alpha}))^{a_{st}}\right] \cdot \sqrt{n}^{\sum_{s \nsim t} a_{st}} \cdot \left(\frac{n}{h}\right)^{\Sigma_t b_t}.$$

STEP 3: Fix $\rho > 0$, let n be large and $\boldsymbol{\alpha} \in B_\theta(n,\rho)$, and approximate

$$\binom{n}{\boldsymbol{\alpha}} = \frac{n!}{\prod_t \alpha_t!}$$

by Stirling's formula as follows:

$$\binom{n}{\boldsymbol{\alpha}} \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{h-1} \cdot \frac{n^{n+\frac{1}{2}}}{\prod_{t=1}^{h} \alpha_t^{\alpha_t + \frac{1}{2}}}.$$

Now

$$\alpha_t = \frac{n}{h}\left(1 + \frac{c_t \cdot h}{\sqrt{n}}\right),$$

so

$$\prod_{t=1}^{h} \alpha_t^{\alpha_t + \frac{1}{2}} = \left(\frac{n}{h}\right)^{\sum_t \alpha_t + h/2} \cdot \prod_t \left(1 + \frac{c_t h}{\sqrt{n}}\right)^{\frac{n}{h} + c_t\sqrt{n} + \frac{1}{2}}.$$

Clearly, the $\frac{1}{2}$ on the right can be discarded. Thus

$$\binom{n}{\boldsymbol{\alpha}} \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{h-1} \left(\frac{1}{n}\right)^{\frac{h-1}{2}} \cdot h^{n+h/2} \cdot \frac{1}{Q},$$

where

$$Q = \prod_{t=1}^{h} \left(1 + \frac{c_t h}{\sqrt{n}}\right)^{\frac{n}{h} + c_t\sqrt{n}}.$$

Let $\ln = \log_e$; then

$$\ln(Q) = \sum_t \left(\frac{n}{h} + c_t\sqrt{n}\right) \ln\left(1 + \frac{c_t h}{\sqrt{n}}\right).$$

Expand $\ln(1 + x) = x - x^2/2 + x^3/3 - + \cdots$ (if $|x| < 1$).

Multiplying and summing over t (note that $c_1 + \cdots + c_h = 0$) we deduce that

$$\ln(Q) = \frac{h}{2}(c_1^2 + \cdots + c_h^2) + 0\left(\frac{1}{\sqrt{n}}\right),$$

hence

$$\frac{1}{Q} \simeq e^{-\frac{h}{2}(c_1^2+\cdots+c_h^2)}$$

and we conclude:

Conclusion: Let $\boldsymbol{\alpha} \in B_\theta(n,\rho)$, $n \to \infty$; then

$$\binom{n}{\boldsymbol{\alpha}} \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{h-1} \cdot h^{h/2} \cdot \left(\frac{1}{n}\right)^{\frac{h-1}{2}} \cdot e^{-\frac{h}{2}(c_1^2+\cdots+c_h^2)} \cdot h^n. \tag{1.2.1}$$

Hence, by Step 2,

$$f_1(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta} \simeq A_1 \cdot A_2 \cdot n^w \cdot h^{\beta n}, \tag{1.2.2}$$

where

$$A_1 = \left(\frac{1}{\sqrt{2\pi}}\right)^{\beta(h-1)} \cdot h^{\frac{\beta h}{2} - \sum_{t=1}^{h} b_t},$$

$$A_2 = A_2(c) = \prod_{1 \le s \nsim t \le h} (c_s - c_t)^{a_{st}} \cdot \exp\Big(-\frac{\beta h}{2}(c_1(\boldsymbol{\alpha})^2 + \cdots + c_h(\boldsymbol{\alpha})^2)\Big),$$

and

$$w = -\frac{\beta}{2}(h-1) + \frac{1}{2}\sum_{1 \le s \nsim t \le h} a_{st} + \sum_{t=1}^{h} b_t.$$

The dependence of the right hand side on $\boldsymbol{\alpha}$ appears only in $A_2 = A_{\text{pol}} \cdot A_{\text{exp}}$, where

$$A_{\text{pol}} = \prod_{s \nsim t}(c_s - c_t)^{a_{st}} \quad \text{and} \quad A_{\text{exp}} = \exp\left(-\frac{\beta h}{2}(c_1^2 + \cdots + c_n^2)\right).$$

Notice that A_{pol} is polynomial in the c_t's, while A_{exp} has rapid Gaussian decay in the c_t's. Thus, by a standard argument (like the classical proof of the Central Limit Theorem of Probability), it follows that

$$\lim_{\rho\to\infty}\left[\lim_{n\to\infty} \frac{\text{Sum}(n)}{\text{Sum}(n,\rho)}\right] = 1,$$

where

$$\text{Sum}(n) = \sum_{\boldsymbol{\alpha}\in B_\theta(n)} f_1(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta} \quad \text{and} \quad \text{Sum}(n,\rho) = \sum_{\boldsymbol{\alpha}\in B_\theta(n,\rho)} f_1(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta};$$

i.e. the asymptotics of $\mathrm{Sum}(n)$ can be calculated by taking $\lim_{\rho\to\infty}$ in the asymptotics of $\mathrm{Sum}(n,\rho)$. This we do next.

STEP 4: In the notation of Step 3 it clearly follows that

$$\mathrm{Sum}(n,\rho) \simeq A_1 \cdot n^w \cdot h^{\beta n} \cdot \sigma,$$

$c_t = c_t(\boldsymbol{\alpha})$ and

$$\sigma = \sum_{\boldsymbol{\alpha}\in B_\theta(n,\rho)} \prod_{1\le s\nsim t\le h} (c_s - c_t)^{a_{st}} \cdot \exp\left(-\frac{\beta h}{2}\sum_{t=1}^{h} c_t^2\right).$$

Denote $\delta_i = \delta_i(\boldsymbol{\alpha}) = c_{r_1+\cdots+r_{i-1}+1} = \cdots = c_{r_1+\cdots+r_i}$, $1\le i\le p$ and

$$\Delta_p(n,\rho) = \left\{(\delta_1,\ldots,\delta_p)\ \middle|\ \text{all } |\delta_i|<\rho \text{ and } \frac{n}{h}+\delta_i\cdot\sqrt{n}\in\mathbb{N} \text{ and } \sum_{i=1}^{p} r_i\delta_i = 0\right\}.$$

Notice that $\boldsymbol{\alpha}\to\delta(\boldsymbol{\alpha}) = (\delta_1(\boldsymbol{\alpha}),\ldots,\delta_p(\boldsymbol{\alpha}))$ is a bijection from $B_\theta(n,\rho)$ onto $\Delta_p(n,\rho)$. Also, $\sum_{t=1}^h c_t^2 = \sum_{i=1}^p r_i\delta_i^2$ and

$$\prod_{1\le s\nsim t\le h} (c_s-c_t)^{a_{st}} = \prod_{1\le i\ne j\le p} (\delta_i-\delta_j)^{e_{ij}}, \quad \text{where } e_{ij} = \sum_{s\in\theta_i, t\in\theta_j} a_{st}.$$

Thus

$$\sigma = \sum_{\underline{\delta}\in\Delta_p(n,\rho)} \prod_{1\le i<j\le p} (\delta_i-\delta_j)^{e_{ij}} \cdot \exp\left(-\frac{\beta h}{2}\sum_{i=1}^{p} r_i\delta_i^2\right).$$

Since $\sum_{i=1}^p r_i\delta_i = 0$, the above sum in the exponential is a $(p-1)$ fold summation. Approximating σ by an integral expression (see, for example, [7, p. 127]) we obtain $\sigma \simeq \sqrt{n}^{p-1}\cdot I'(\rho)$, where

$$I'(\rho) = \underset{\substack{r_1x_1+\cdots+r_px_p=0\\ |x_1|,\ldots,|x_p|<\rho}}{\int\cdots\cdots\int} \prod_{1\le i\ne j\le p} (x_i-x_j)^{e_{ij}} \exp\left(-\frac{\beta h}{2}\sum_{i=1}^{p} r_ix_i^2\right) d^{(p-1)}x.$$

Conclusions: In the previous notations

$$\mathrm{Sum}(n,\rho) \simeq A_1\cdot I(\rho)\cdot n^{w+\frac{p-1}{2}}\cdot h^{\beta n}.$$

Taking $\lim_{\rho\to\infty}$ we obtain $\mathrm{Sum}(n) \simeq A_1\cdot I'\cdot n^{w+\frac{p-1}{2}}\cdot h^{\beta n}$, where

$$I' = \underset{r_1x_1+\cdots+r_px_p=0}{\int\cdots\cdots\int} \prod_{1\le i\ne j\le p} (x_i-x_j)^{e_{ij}} \exp\left(-\frac{\beta h}{2}\sum_{i=1}^{p} r_ix_i^2\right) d^{(p-1)}x.$$

Finally, make a change of variables $u_i = \sqrt{h}\, x_i$ in I'. Clearly, I' is transformed into I, while the factor

$$\left(\frac{1}{\sqrt{h}}\right)^{\sum e_{ij}+p-1} = h^{-\frac{1}{2}\sum_{s\nsim t} a_{st}-\frac{1}{2}(p-1)}$$

now multiplies the previous constant

$$A\cdot d = \left(\frac{1}{\sqrt{2\pi}}\right)^{\beta(h-1)} \cdot h^{\frac{\beta h}{2}-\sum_{t=1}^{h} b_t} \cdot d, \quad d = \prod_{s\sim t}\prod_{v=1}^{a_{st}} d_{stv}. \quad \blacksquare$$

2. Transition to d_λ

Let $\theta, B_\theta(n)$ and $f(\boldsymbol{\alpha})$ be as in 1.1; define

$$\Lambda_\theta(n) = B_\theta(n)\cap \mathrm{Par}(n) = \{\boldsymbol{\alpha}\in B_\theta(n) \,\big|\, \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_h\},$$

and consider the asymptotics of

$$\sum_{\boldsymbol{\alpha}\in\Lambda_\theta(n)} f(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta}.$$

The previous calculations lead essentially to the same result, the only difference being that the domain of integration now has the extra condition $x_1 \geq \cdots \geq x_p$. Thus

THEOREM 2.1: *Let $B_\theta(n)$ and $f(\boldsymbol{\alpha})$ be as in 1.1 and $\Lambda_\theta(n) = B_\theta(n)\cap\mathrm{Par}(n)$. Then $\sum_{\boldsymbol{\alpha}\in\Lambda_\theta(n)} f(\boldsymbol{\alpha})\binom{n}{\boldsymbol{\alpha}}^{\beta} \simeq c\cdot I_1\cdot n^u\cdot h^{\beta n}$, where c and u are given in Theorem 1.2 and where*

$$I_1 = \int_{\substack{r_1x_1+\cdots+r_px_p=0\\ x_1\geq\cdots\geq x_p}} \cdots\cdots\int \prod_{1\leq i\neq j\leq p}(x_i-x_j)^{e_{ij}}\cdot\exp\left(-\frac{\beta}{2}\sum_{i=1}^{p} r_i x_i^2\right) d^{(p-1)}x.$$

Recall

$$e_{ij} = \sum_{s\in\theta_i, t\in\theta_j} a_{st}, \quad c = \left(\frac{1}{\sqrt{2\pi}}\right)^{\beta(h-1)}\cdot h^{-u+\frac{\beta}{2}}\cdot d,$$

$$d = \prod_{s\sim t}\prod_{v=1}^{a_{st}} d_{stv}, \quad u = -\frac{\beta h}{2}+\frac{1}{2}\sum_{1\leq s\nsim t\leq h} a_{st} + \sum_{t=1}^{h} b_t + \frac{1}{2}(p-1)+\frac{\beta}{2}.$$

Next, we calculate the asymptotics of $\sum_{\lambda\in\Lambda_\theta(n)} f(\lambda) d_\lambda^\beta$, where d_λ denotes the number of standard Young tableaux of shape λ.

THEOREM 2.2: *Let $a_{st}, a_t \in \mathbb{N}$ and let*

$$f(\lambda) = \left[\prod_{1\le s<t\le h}\prod_{v=1}^{a_{st}}(\lambda_s - \lambda_t + d_{stv})\right]\cdot\left[\prod_{t=1}^{h}\prod_{v=1}^{a_t}(\lambda_t + d_{tv})\right],$$

with d_{stv} and d_{tv} constants and $\Lambda_\theta(n)$ as above. Then

$$\sum_{\lambda\in\Lambda_\theta(n)} f(\lambda)d_\lambda^\beta \simeq c_2\cdot I_2\cdot n^{u_2}\cdot h^{\beta n},$$

where

$$u_2 = -\frac{\beta h^2}{4} + \frac{1}{2}\sum_{\substack{1\le s<t\le h\\ s\not\sim t}} a_{st} + \sum_{t=1}^{h} a_t - \frac{\beta}{4}\sum_{i=1}^{p} r_i^2 + \frac{1}{2}(p-1) + \frac{\beta}{2},$$

$$c_2 = \left(\frac{1}{\sqrt{2\pi}}\right)^{\beta(h-1)}\cdot h^{-u_2+\frac{\beta}{2}}\cdot d',\quad d' = \prod_{\substack{1\le s<t\le h\\ s\sim t}}\left[(t-s)^\beta\prod_{v=1}^{a_{st}} d_{stv}\right]$$

and

$$I_2 = \int_{\substack{r_1x_1+\cdots+r_px_p=0\\ x_1\ge\cdots\ge x_p}}\cdots\cdots\int \prod_{1\le i<j\le p}(x_i - x_j)^{e_{ij}+\beta r_i r_j}\cdot\exp\left(-\frac{\beta}{2}\sum_{i=1}^{p} r_i x_i^2\right) d^{(p-1)}x$$

(again, $e_{ij} = \sum_{s\in\theta_i, t\in\theta_j} a_{st}$).

Proof: By the Young Frobenius formula,

$$d_\lambda = \binom{n}{\lambda}\frac{\prod_{1\le s<t\le h}(\lambda_s - \lambda_t + t - s)}{\prod_{s=1}^{h-1}\prod_{t=1}^{h-s}(\lambda_s + t)},$$

hence $f(\lambda)d_\lambda^\beta = g(\lambda)\binom{n}{\lambda}^\beta$, where

$$g(\lambda) = f(\lambda)\cdot\prod_{1\le s<t\le h}(\lambda_s - \lambda_t + t - s)^\beta\cdot\prod_{s=1}^{h-1}\prod_{t=1}^{h-s}(\lambda_s + t)^{-\beta} = M_1\cdot M_2,$$

with

$$M_1 = \prod_{1\le s<t\le h}\left[(\lambda_s - \lambda_t + t - s)^\beta\cdot\prod_{v=1}^{a_{st}}(\lambda_s - \lambda_t + d_{stv})\right]$$

and

$$M_2 = \prod_{s=1}^{h-1}\prod_{t=1}^{h-s}(\lambda_s + t)^{-\beta}\cdot\prod_{s=1}^{h}\cdot\prod_{v=1}^{a_s}(\lambda_s + d_{sv}).$$

Rearranging terms, we can write

$$M_1 = \prod_{1\le s<t\le h} \prod_{v=1}^{a'_{st}} (\lambda_s - \lambda_t + d'_{stv}) \quad \text{and} \quad M_2 = \prod_{s=1}^{h} \prod_{v=1}^{a'_s} (\lambda_s + d'_{sv})^{\varepsilon_{sv}}.$$

Here $a'_{st} = a_{st} + \beta$,

$$d'_{stv}, = \begin{cases} d_{stv}, & 1 \le v \le a_{st}, \\ t-s, & a_{st}+1 \le v \le a_{st}+\beta, \end{cases}$$

$b_s = \sum_{v=1}^{a_s} \varepsilon_{sv} = a_s$, $a'_s = a_s + \beta(h-s)$ and $b'_s = \sum_{v=1}^{a'_s} \varepsilon_{sv} = a_s - \beta(h-s)$.

Applying Theorem 2.1 we have

$$\sum_{\lambda\in\Lambda_\theta(n)} f(\lambda) d_\lambda^\beta = \sum_{\lambda\in\Lambda_\theta(n)} g(\lambda) \binom{n}{\lambda}^\beta \underset{n\to\infty}{\simeq} c_2 I_2 n^{u_2} h^{\beta n},$$

where

$$u_2 = -\frac{\beta h}{2} + \frac{1}{2} \sum_{\substack{1\le s<t\le h \\ s\nsim t}} a'_{st} + \sum_{t=1}^{h} b'_t + \frac{1}{2}(p-1) + \frac{\beta}{2},$$

$$c_2 = \left(\frac{1}{\sqrt{2\pi}}\right)^{\beta(h-1)} \cdot h^{-u_2+\frac{\beta}{2}} \cdot d',$$

$$d' = \prod_{\substack{1\le s<t\le h \\ s\sim t}} \left[\prod_{v=1}^{a'_{st}} d'_{stv}\right] = \prod_{\substack{1\le s<t\le h \\ s\sim t}} \left[(t-s)^\beta \prod_{v=1}^{a_{st}} d_{stv}\right]$$

and

$$I_2 = \underset{\substack{r_1x_1+\cdots+r_px_p=0 \\ x_1\ge\cdots\ge x_p}}{\int\cdots\cdots\int} \prod_{1\le i<j\le p} (x_i - x_j)^{e'_{ij}} \cdot \exp\left(-\frac{\beta}{2}\sum_{i=1}^{p} r_i x_i^2\right) d^{(p-1)}x.$$

Here $e'_{ij} = \sum_{s\in\theta_i, t\in\theta_j} a'_{st}$.

Simplify u_2 first: $a'_{st} = a_{st} + \beta$, and

$$\sum_{\substack{1\le s<t\le h \\ s\nsim t}} 1 = \sum_{1\le s<t\le h} 1 - \sum_{\substack{1\le s<t\le h \\ s\sim t}} 1 = \frac{1}{2}h(h-1) - \sum_{i=1}^{p} \frac{1}{2} r_i(r_i - 1)$$

$$\left(\text{since} \sum_{\substack{s<t \\ s\sim t}} 1 = \sum_{i=1}^{p} \sum_{\substack{s,t\in\theta_i \\ s<t}} 1\right), \text{ hence}$$

$$\sum_{\substack{1\le s<t\le h \\ s\nsim t}} a'_{st} = \sum_{\substack{1\le s<t\le h \\ s\nsim t}} a_{st} + \frac{\beta}{2} h(h-1) - \frac{\beta}{2} \sum_{i=1}^{p} r_i(r_i - 1).$$

Also, $b'_t = a_t - \beta(h-t)$, hence

$$\sum_{t=1}^{h} b'_t = \sum_{t=1}^{h} a_t - \frac{\beta h(h-1)}{2}.$$

Note also that $\sum_{i=1}^{p} r_i(r_i - 1) = \sum_{i=1}^{p} r_i^2 - h$. Thus

$$u_2 = -\frac{\beta h^2}{4} + \frac{1}{2} \sum_{\substack{1 \le s < t \le h \\ s \not\sim t}} a_{st} + \sum_{t=1}^{h} a_t - \frac{\beta}{4} \sum_{i=1}^{p} r_i^2 + \frac{1}{2}(p-1) + \frac{\beta}{2}.$$

Finally,

$$e'_{ij} = \sum_{s\in\theta_i, t\in\theta_j} a'_{st} = \sum_{s\in\theta_i, t\in\theta_j} (a_{st} + \beta) = e_{ij} + \beta r_i r_j,$$

since $e_{ij} = \sum_{s\in\theta_i, t\in\theta_j} a_{st}$.

Note that u_2 in Theorem 2.2 can also be written as

$$u_2 = -\frac{1}{2}\beta(h^2 - 1) + \frac{1}{2} \sum_{\substack{1 \le s < t \le h \\ s \not\sim t}} (a_{st} + \beta) + \sum_{s=1}^{h} a_s + \frac{1}{2}(p-1).$$

3. Degrees of Young derived sequences

Let $\eta^q = \{\eta_n^q\}_{n\ge 0}$ denote the S_n-character sequence obtained from an S_n-character $\eta = \{\eta_n\}_{n\ge 0}$ by a q-column dilation of the Young diagrams, and let $y^l(\eta^q) = \{y^l(\eta^q)_n\}_{n\ge 0}$ denote its lth Young derived sequence as defined in [8] (see also the Introduction). In [2] a formula, valid for $q \ge 1$ and $0 \le l \le q-1$, was given expressing the coefficients of the irreducible characters in $y^l(\eta^q)$ in terms of those in η and of semi-standard Young tableaux.

If the Young diagrams of η are of height k, then the Young diagrams of $y^l(\eta^q)$ are of height $qk + l$. Let $a_{ij}, 1 \le i < j \le k$ be integers and let $F = F(x)$ be the function on $\mathbb{R}^k$ given by

$$F(x) = F(x_1, \ldots, x_k) = \prod_{1\le i<j\le k} (x_i - x_j)^{a_{ij}}.$$

Let

$$\eta_n = \sum_{\lambda\in\Lambda_k(n)} F(\lambda)\chi_\lambda \quad (\Lambda_k(n) = \{\lambda \in \operatorname{Par}(n) \mid \lambda_{k+1} = 0\}),$$

where χ_λ is the irreducible S_n-character associated to λ. By [2, Theorem 1.2], if $0 \le l \le q-1$ then

$$y^l(\eta^q)_n = \sum_{\mu\in\Lambda_\theta(n)} b^{(l)}(\mu)\chi_\mu,$$

where θ is determined, as in previous sections, by specifying integers $r_1, \dots, r_p$ summing to $qk + l$. Here $p = kl + k + l$ and

$$r_i = \begin{cases} 1, & i = m(l+1)+1, \dots, m(l+1)+l;\ m = 0, \dots, k, \\ q - l & i = m(l+1);\ m = 1, \dots, k. \end{cases}$$

The formula for $b^{(l)}(\mu) = b^{(l)}(\mu_1, \dots, \mu_{qk+l})$ is given by

$$b^{(l)}(\mu) = F(\mu_q, \dots, \mu_{kq})\{1!2!\cdots(l-1)!\}^{-(k+1)} \prod_{m=0}^{k} \prod_{mq+1 \le s < t \le mq+l} (\mu_s - \mu_t + t - s).$$

The function $b^{(l)}(\mu)$ is of the type that can be handled using Theorem 2.2 of the present paper. In particular, we shall show the following

PROPOSITION 3.1: *With the above notation we have*

$$\deg\ y^l(\eta^q)_n \underset{n\mapsto\infty}{\simeq} C \cdot I \cdot n^u \cdot (qk+l)^{n-u+\frac{1}{2}}$$

where

$$C = \Big(\frac{1}{\sqrt{2\pi}}\Big)^{(qk+l-1)} \{1!2!\cdots(q-l-1)!\}^k \{1!2!\cdots(l-1)!\}^{-(k+1)},$$

$$u = \frac{1}{2} \sum_{1 \le i < j \le k} a_{ij} - \frac{1}{4} k q^2 (k+1) + \frac{1}{2} k$$

and

$$I = \int \cdots \int_{\mathfrak{R}_{k,l,q}} V(x_1, \dots, x_{kl+k+l})\ \exp\Big(-\frac{1}{2}\|x\|^2_{k,l,q}\Big)\ d^{(kl+k+l-1)}x$$

with

$$\mathfrak{R}_{k,l,q} = \{x_1 \ge \cdots \ge x_{kl+k+l} \mid \sum_{m=0}^{k} \sum_{i=m(l+1)+1}^{m(l+1)+l} x_i + (q-l) \sum_{m=1}^{k} x_{m(l+1)} = 0\},$$

$$V(x) = \prod_{m=0}^{k} \prod_{\{m(l+1)+1 \le i < j \le m(l+1)+l\}} (x_i - x_j)$$
$$\times \prod_{1 \le i < j \le k} (x_{i(l+1)} - x_{j(l+1)})^{a_{ij}} \prod_{1 \le i < j \le kl+k+l} (x_i - x_j)^{r_i r_j}$$

and

$$\|x\|^2_{k,l,q} = \sum_{m=0}^{k} \sum_{i=m(l+1)+1}^{m(l+1)+l} x_i^2 + (q-l) \sum_{m=1}^{k} x^2_{m(l+1)}.$$

Before proving Proposition 3.1, we give the application to identities between multi-integrals. In [2] the asymptotics of deg $y^l(\eta^q)_n$ was computed in another way from deg η_n and general results about Young derived sequences. This also leads to a multi-integral expression, but of a different form to that of Proposition 3.1. Equating the two asymptotics leads to an identity between multi-integrals. This was carried out for the case $q = l - 1$ in [2, Theorem 4]. We obtain the following generalization of that result.

THEOREM 3.2: *With the above notation and letting $D_k = D_k(x)$ be the function on $\mathbb{R}^k$ given by*

$$D_k(x) = \prod_{1 \le i < j \le k} (x_i - x_j),$$

let I' be the multi-integral expression

$$I' = \int \cdots \int_{S_k} F(x)(D_k(x))^{q^2} \exp(-\frac{q}{2}\sum_{i=1}^{k} x_i^2) d^{(k-1)}x$$

where

$$S_k = \{x_1 \ge \cdots \ge x_k \mid \sum_{i=1}^{k} x_i = 0\}.$$

Let I be the multi-integral expression of Proposition 3.1, so that

$$I = \int \cdots \int_{\mathfrak{R}_{k,l,q}} V(x_1, \ldots, x_{kl+k+l}) \ \exp\left(-\frac{1}{2}\|x\|^2_{k,l,q}\right) \ d^{(kl+k+l-1)}x.$$

Then we have

$$I' = CI$$

where

$$C' = \left(\frac{1}{\sqrt{2\pi}}\right)^l \left(\frac{q}{k}\right)^{\frac{1}{2}} \sqrt{qk + l}\{1!2! \cdots (l-1)!\}^{-(k+1)}$$
$$\times \{1!2! \cdots (q-l-1)!\}^k \{1!2! \cdots (q-1)!\}^{-k}.$$

Notice that the constant C' in the above theorem does not depend on the function F.

If we substitute $F(x) = D_k(x)^p$ in Theorem 3.2, we obtain

THEOREM 3.3: *In the notation of Theorem 3.2, let $F(x) = D_k(x)^p$. Then*

$$I' = C'I,$$

where

$$I' = \int \cdots \int_{S_k} D_k(x)^{p+q^2} \cdot \exp\left(-\frac{q}{2}\sum_{i=1}^{k} x_i^2\right) d^{(k-1)}x,$$

C' and I are as in Theorem 3.2, and here

$$V(x) = \prod_{m=0}^{k} \prod_{\{m(l+1)+1\le i<j\le m(l+1)+l\}} (x_i - x_j)$$
$$\times \prod_{1\le i<j\le k} (x_{i(l+1)} - x_{j(l+1)})^p \prod_{1\le i<j\le kl+k+l} (x_i - x_j)^{r_i r_j}$$

with the r_i as before.

NOTE: I' in Theorem 3.3 is a Mehta integral, hence it can be evaluated, which then implies the evaluation of the multi-integral I.

We proceed now to prove Proposition 3.1 and Theorem 3.2.

Proof of Proposition 3.1: We apply [2, Theorem 1.2], Theorem 2.2 and "deg" to

$$\{1!2!\cdots(l-1)!\}^{(k+1)} y^l (\eta^q)_n = \sum_{\mu\in\Lambda_\theta(n)} f(\mu)\chi_\mu$$

where, of course,

$$f(\mu) = F(\mu_q,\ldots,\mu_{kq}) \prod_{m=0}^{k} \prod_{\{mq+1\le s<t\le mq+l\}} (\mu_s - \mu_t + t - s).$$

We are dealing with partitions of height $h = qk + l$ and in order to apply Theorem 2.2 we rewrite $f(\mu)$ in the form

$$f(\mu) = \prod_{1\le s<t\le qk+l} (\mu_s - \mu_t + d_{st})^{b_{st}}$$

and calculate the b_{st} and d_{st}. Then Theorem 2.2 (with $\beta = 1$) gives a result of the form

$$\{1!2!\cdots(l-1)!\}^{(k+1)} \deg\ y^l(\eta^q)_n \underset{n\to\infty}{\simeq} c\cdot I\cdot n^u\cdot (qk+l)^{n-u+\frac{1}{2}}.$$

We have to show that this agrees with the formula of the proposition. The constant c is given by

$$c = \left(\frac{1}{\sqrt{2\pi}}\right)^{(qk+l-1)} \prod_{s\sim t} d_{st}^{b_{st}} \prod_{s\sim t}(t-s).$$

Here $1 \le s < t \le p = (kl + k + l)$ and $s \sim t$ is the equivalence relation with respect to the θ of the proposition, namely that determined by the integers

$$r_i = \begin{cases} 1, & i = m(l+1)+1,\ldots,m(l+1)+l;\ m = 0,\ldots,k, \\ q-l, & i = m(l+1);\ m = 1,\ldots,k. \end{cases}$$

We have

$$u = -\frac{1}{2}((qk+l)^2 - 1) + \frac{1}{2}\sum_{s\not\sim t}(b_{st}+1) + \frac{1}{2}(kl+k+l-1)$$

and

$$I = \int\cdots\int_{\mathfrak{R}} \prod_{1\le i<j\le p}(x_i - x_j)^{r_{ij}} \exp(-\frac{1}{2}(r_1x_1^2 + \cdots + r_px_p^2))d^{(p-1)}x$$

where, for $1 \le i < j \le p$,

$$r_{ij} = \sum_{s\in\theta_i}\sum_{t\in\theta_j}(b_{st}+1).$$

The region $\mathfrak{R}$ in $\mathbb{R}^{p-1}$ is

$$\mathfrak{R} = \{x_1 \ge \cdots \ge x_p |\, r_1x_1 + \cdots + r_px_p = 0\}.$$

Now by the definition of F and the formula for $f(\mu)$ we have

$$\prod_{1\le s<t\le qk+l}(\mu_s - \mu_t + d_{st})^{b_{st}} = \prod_{1\le i<j\le k}(\mu_{iq} - \mu_{jq})^{a_{ij}} \prod_{m=0}^{k} \prod_{\substack{mq+1\le s\\ <t\le mq+l}} (\mu_s - \mu_t + t - s).$$

Comparing both sides we have

$$b_{st} = \begin{cases} a_{ij}, & s = iq, t = jq;\ 1 \le i < j \le k, \\ 1, & mq+1 \le s < t \le mq+l;\ m = 0,\ldots,k. \end{cases}$$

Hence

$$\sum_{s\in\theta_i}\sum_{t\in\theta_j} b_{st} = \begin{cases} a_{i'j'}, & i = i'(l+1), j = j'(l+1);\ 1 \le i' < j' \le k, \\ 1, & m(l+1)+1 \le i < j \le m(l+1)+l;\ m = 0,\ldots,k, \end{cases}$$

so that

$$\prod_{1\le i<j\le p}(x_i - x_j)^{r_{ij}} = \prod_{m=0}^{k} \prod_{\{m(l+1)+1\le i<j\le m(l+1)+l\}} (x_i - x_j)$$
$$\times \prod_{1\le i<j\le k}(x_{i(l+1)} - x_{j(l+1)})^{a_{ij}} \prod_{1\le i<j\le kl+k+l}(x_i - x_j)^{r_ir_j}.$$

This is the function $V(x)$ of the proposition. We now turn to the computation of u. By inspection of the formula for b_{st} we have

$$\sum_{s\not\sim t} b_{st} = \sum_{1\le i<j\le k} a_{ij} + \sum_{m=0}^{k} \sum_{\{mq+1\le s<t\le mq+l\}} 1$$
$$= \sum_{1\le i<j\le k} a_{ij} + \frac{1}{2}(k+1)l(l-1).$$

Viewing the s,t with $s \not\sim t$ as the complement of the s,t with $s \sim t$ gives

$$\sum_{s \not\sim t} 1 = \frac{1}{2}(qk+l)(qk+l-1) - \frac{1}{2}k(q-l)(q-l-1).$$

These formulae sum to give

$$\sum_{s \not\sim t} (b_{st}+1)$$

and on substituting into u and simplifying we have finally

$$u = \frac{1}{2} \sum_{1 \le i < j \le k} a_{ij} - \frac{1}{4}kq^2(k+1) + \frac{1}{2}k,$$

which is the expression of the proposition.

Looking now at the expression for the constant c, we observe that

$$\prod_{s \sim t} d_{st}^{b_{st}} = 1$$

and

$$\prod_{s \sim t} (t-s) = \{1!2! \cdots (q-l-1)!\}^k,$$

so that we obtain the expression of the proposition for C since

$$C = \{1!2! \cdots (l-1)!\}^{-(k+1)} c.$$

To complete the proof, we observe that $\mathfrak{R} = \mathfrak{R}_{k,l,q}$ and that the integral I we have derived agrees with the integral I of the statement of the proposition. ∎

The proof of Theorem 3.2 will follow easily once we remark that by the change of variables

$$x_i \mapsto \frac{1}{\sqrt{k}} x_i, \quad i = 1 \ldots, k$$

and by some routine simplifications, the result of [2, Theorem 4.1] (with a correction: the $\sqrt{k}$ in c of [2, Thm. 4.1] should have been $\sqrt{q}$), which gives the alternate computation of deg $y^l(\eta^q)_n$, may be restated as follows.

PROPOSITION 3.4 (2, Thm. 4.1): *With the above notation we have*

$$\deg\ y^l(\eta^q)_n \underset{n \mapsto \infty}{\simeq} c' \cdot I' \cdot n^u \cdot (qk+l)^{n-u}$$

where

$$u = \frac{1}{2} \sum_{1 \le i < j \le k} a_{ij} - \frac{1}{4}kq^2(k+1) + \frac{1}{2}k,$$

$$c' = \left(\frac{1}{\sqrt{2\pi}}\right)^{qk-1} \sqrt{\frac{k}{q}}\{1!2!\cdots(q-1)!\}^k$$

and I' is the multi-integral expression of Theorem 3.2.

Proof of Theorem 3.2: Equating the asymptotics of deg $y^l(\eta^q)_n$ given on the one hand in Proposition 3.1 and on the other in Proposition 3.4, we have

$$C \cdot I \cdot n^u \cdot (qk+l)^{n-u+\frac{1}{2}} = c' \cdot I' \cdot n^u \cdot (qk+l)^{n-u}$$

so that

$$C' = C \cdot (c')^{-1} \cdot \sqrt{qk+l},$$

which on simplifying gives

$$C' = \left(\frac{1}{\sqrt{2\pi}}\right)^l \left(\frac{q}{k}\right)^{\frac{1}{2}} \sqrt{qk+l}\{1!2!\cdots(l-1)!\}^{-(k+1)} \times \{1!2!\cdots(q-l-1)!\}^k\{1!2!\cdots(q-1)!\}^{-k}$$

as required. ∎

4. "Nice" polynomials

Theorem 3.7 of [8] was proved for functions satisfying a "niceness" property. Here we show that the "p.h.d." polynomials defined below do satisfy that property, hence that "Theorem 3.7" applies.

Definition 4.1: The polynomial $a(x_1,\ldots,x_h)$ is called p.h.d. if it satisfies the following three properties:

(p) If $\lambda_1 > \cdots > \lambda_h$ then $a(\lambda_1,\ldots,\lambda_h) > 0$ (*positive*),

(h) $a(x)$ is homogeneous (*homogeneous*),

(d) $a(x)$ is a polynomial of the differences $x_i - x_j$'s: $a(x_1+s,\ldots,x_h+s) = a(x_1,\ldots,x_h)$ for all s (*differences*).

Definition 4.2: Partition summation and partition integration:

For $\mu = (\mu_1,\ldots,\mu_h) \in \mathbb{R}^h$ and $a = a(x_1,\ldots,x_h)$ a function on $\mathbb{R}^h$, define

$$p \cdot i(\mu;\ a) = \int_{\mu_2}^{\mu_1} dx_1 \cdots \int_{\mu_{h+1}}^{\mu_h} dx_h \cdot a(x_1,\ldots,x_h).$$

Assume further that μ is a partition; then define

$$p \cdot s(\mu;\ a) = \sum_{\lambda_1=\mu_2}^{\mu_1} \cdots \sum_{\lambda_h=\mu_{h+1}}^{\mu_h} a(\lambda_1,\ldots,\lambda_h).$$

The discussion in 4.6–4.10 below shows that in the sense of [8, 3.6] p.h.d. polynomials are "nice": $p \cdot i(\mu, a)$ approximates $p \cdot s(\mu, a)$. Hence we can apply Theorem 3.7 of [8] and essentially restate that theorem as:

THEOREM 4.3 (8, Thm. 3.7): *Let*

$$VGI_h(a) \stackrel{\text{def}}{=\!=} \int_{\substack{x_1+\cdots+x_h=0 \\ x_1\geq\cdots\geq x_h}} \cdots \int a(x_1,\ldots,x_h) \prod_{i\leq i<j\leq h} (x_i - x_j)\cdot \exp\left(-\frac{h}{2}\sum_{i=1}^{h} x_i^2\right) d^{(h-1)}x$$

(V for Vandermonde, G for Gaussian measure, I for integration).

Let a polynomial $a = a(x_1, \cdots, x_h)$ be p.h.d. of degree d, and

$$z = (z_1, \ldots, z_{h+1}) \in \mathbb{R}^{h+1}.$$

Let

$$p = p(z_1, \ldots, z_{h+1}) = p \cdot i(z;\ a).$$

Then

$$VGI_{h+1}(p) = c(h) \cdot VGI_h(a),$$

where

$$c(h) = \left(\frac{h}{h+1}\right)^{\frac{1}{2}(d+h-(\frac{1}{2})h(h-1))} \sqrt{\frac{2\pi}{h+1}}.$$

More generally, given $a = a(x) = a(x_1, \ldots, x_h)$, define the sequence of polynomials $a^{(s)}$ in $s + h$ variables by induction:

$$a^{(0)} = a^{(0)}(x_1, \ldots, x_h) = a(x_1, \ldots, x_h)$$

and

$$a^{(s+1)} = a^{(s+1)}(z_1, \ldots, z_{h+s+1}) = p \cdot i((z_1, \ldots, z_{h+s+1});\ a^{(s)}).$$

If $a(x)$ is p.h.d. then

$$VGI_{h+s}(a^{(s)}) = \left(\prod_{j=0}^{s-1} c(h+j)\right) \cdot VGI_h(a),$$

i.e.

$$\int_{\substack{x_1+\cdots+x_{h+s}=0 \\ x_1\geq\cdots\geq x_{h+s}}} \cdots \int a^{(s)}(x_1, \ldots, x_{h+s}) \cdot \left(\prod_{1\leq i<j\leq h+s} (x_i - x_j)\right)$$

$$\cdot \exp\left(-\frac{h+s}{2}\sum_{i=1}^{h+s} x_i^2\right) d^{(h+s-1)}x =$$

$$\left(\prod_{j=0}^{s-1} c(h+j)\right) \cdot \int_{\substack{x_1+\cdots+x_h=0\\ x_1\geq\cdots\geq x_h}} \cdots \int a(x_1,\ldots,x_h)$$
$$\cdot \prod_{1\leq i<j\leq h} (x_i-x_j)\cdot \exp\left(-\frac{h}{2}\sum_{i=1}^{h} x_i^2\right) d^{(h-1)}x.$$

Remark 4.4: The discussion in 4.6–4.10 also yields an approximation solution to the following problem:

Let $\eta_n = \sum_{\lambda\in\Lambda_k(n)} F(\lambda)\chi_\lambda$ and write

$$(y^\ell(\eta^q))_n = \sum_{\mu\in\Lambda_{qk+\ell}(n)} b^{(\ell)}(\mu)\chi_\mu.$$

The multiplicities $b^{(\ell)}(\mu)$ were calculated only for $0\leq \ell\leq q-1$ ([2], Thm. 1.2). The problem of calculating the $b^{(\ell)}$'s for $q\leq\ell$ is open. However, we shall prove

THEOREM 4.5: *Let $F(x)=\prod_{1\leq i<j\leq k}(x_i-x_j)^{a_{ij}}$ with $b^{(\ell)}(\mu)$ as in 4.4.*
Define

$$a^{(0)}(x) = F(x_q, x_{2q},\ldots,x_{kq})\cdot\left(\prod_{t=0}^{k}\ \prod_{tq+1\leq i<j\leq tq+q-1}(x_i-x_j)\right)$$
$$\cdot\,(1!2!\cdots(q-2)!)^{-k-1},$$

then construct $\{a^{(s)}\}_{s\geq 0}$ inductively as in 4.3:

$$a^{(s+1)}(z) = p\cdot i(z;\ a^{(s)}).$$

Then:

(4.5.1) *If $q\leq\ell$ then $b^{(\ell)}(\mu)$ is a polynomial of the $\mu_i-\mu_j$'s,*

(4.5.2) *$a^{(s)}(x)$ is p.h.d. for all $s\geq 0$, and*

(4.5.3) *$b^{(\ell)}(\mu) = a^{(\ell-q+1)}(\mu)+$ lower terms in the $\mu_i-\mu_j$'s.*

Thus, $b^{(\ell)}$ is approximated by $a^{(\ell-q+1)}$ (in our earlier notation, $b^{(\ell)}(\mu)\approx a^{(\ell-q+1)}(\mu)$).

To establish "niceness", we proceed as follows:

LEMMA 4.6: *Let $a = a(x_1,\ldots,x_h)$ be p.h.d. of degree d, and let $p(z) = p(z_1,\ldots,z_{h+1}) = p\cdot i(z,a)$. Then $p(z)$ is again p.h.d. and of degree $d+h$.*

Proof: By Lemma 3.4 of [8] we only need to check positivity (p):

Let $\mu_1>\cdots>\mu_{h+1}$ and show that $p\cdot i(\mu, a(x)) = p(\mu_1,\ldots,\mu_{h+1})>0$.

Indeed, let $\varepsilon > 0$ satisfy $\mu_i - \varepsilon > \mu_{i+1} + \varepsilon$, $i = 1, \dots, h$. If

$$\mu_i - \varepsilon \geq \lambda_i \geq \mu_{i+1} + \varepsilon, \quad i = 1, \dots, h,$$

then $\lambda_1 > \cdots > \lambda_h$, so $a(\lambda_1, \dots, \lambda_h) > 0$. By the (multi-variables) mean-value theorem, it now follows that for some $(\lambda_1, \dots, \lambda_h)$ with

$$\mu_i - \varepsilon \geq \lambda_i \geq \mu_{i+1} + \varepsilon, \quad i = 1, \dots, h,$$

we have

$$\begin{aligned} p(\mu_1, \dots, \mu_{h+1}) &\geq \int_{\mu_2+\varepsilon}^{\mu_1-\varepsilon} dx_1 \cdots \int_{\mu_{h+1}+\varepsilon}^{\mu_k-\varepsilon} dx_h \cdot a(x_1, \dots, x_h) \\ &= \left(\prod_{i=1}^{h} (\mu_i - \mu_{i+1} - 2\varepsilon) \right) \cdot a(\lambda_1, \dots, \lambda_h) > 0. \quad \blacksquare \end{aligned}$$

Remark 4.7: Let $b_i < d_i \in \mathbb{N}$, $i = 1, \dots, h$ and let $f(x_1, \dots, x_h)$ be a polynomial of degree d.

Let

$$p_1(b, d) = p_1(b_1, \dots, b_h, d_1, \dots, d_h) = \int_{b_1}^{d_1} dx_1 \cdots \int_{b_h}^{d_h} dx_h \cdot f(x_1, \dots, x_h)$$

and

$$p_2(b, d) = p_2(b_1, \dots, b_h, d_1, \dots, d_h) = \sum_{i_1=b_1}^{d_1} \cdots \sum_{i_h=b_h}^{d_h} f(i_1, \dots, i_h).$$

Then both $p_1(b, d)$ and $p_2(b, d)$ are polynomials in $b_1, \dots, b_k, d_1, \dots, d_k$, of total degree $d + h$. Moreover, it is well known that $p_2(b, d) = p_1(b, d) + r(b, d)$, where $r(b_1, \dots, b_h, d_1, \dots, d_h)$ is a polynomial of total degree $\leq d + h - 1$ in $b_1, \dots, b_h, d_1, \dots, d_h$.

COROLLARY 4.8: *Let $f = f(x_1, \dots, x_h)$ be a polynomial of degree d and let $\mu_1 > \cdots > \mu_{h+1}$ (μ is a partition). Then (see 4.2)*

$$p \cdot s(\mu;\ f) = p \cdot i(\mu;\ f) + \bar{r}(\mu),$$

where both sides are polynomials in $\mu_1, \dots, \mu_{h+1}$ of degree $d + h$, but $\bar{r}(\mu)$ is of degree $\leq d + h - 1$.

LEMMA 4.9: *Let $f = f(x_1, \dots, x_h)$ and $\mu = (\mu_1, \dots, \mu_{h+1})$ be as in 4.8, and assume further that f is a polynomial of the $x_i - x_j$'s; then this is also true of $\bar{r}(\mu)$ and $p \cdot s(\mu;\ f)$.*

Proof: By Lemma 3.4 of [8], $p \cdot i(\mu;\ f)$ is a polynomial of the $x_i - x_j$'s, hence it suffices to check that for any $s \in \mathbb{Z}$,

$$p \cdot s((\mu_1 + s, \dots, \mu_{h+1} + s);\ f) = p.s((\mu_1, \dots, \mu_{h+1});\ f).$$

This follows, since

$$\begin{aligned} p \cdot s((\mu_1 + s, \dots, \mu_{h+1} + s); f) &= \sum_{\lambda_1 = \mu_2 + s}^{\mu_1 + s} \cdots \sum_{\lambda_k = \mu_{h+1} + s}^{\mu_h + s} f(\lambda_1, \dots, \lambda_k)|_{(\lambda_i = \eta_i + s)} \\ &= \sum_{\eta_1 = \mu_2}^{\mu_1} \cdots \sum_{\eta_h = \mu_{h+1}}^{\mu_h} f(\eta_1 + s, \dots, \eta_h + s) \\ &= \sum_{\eta_1 = \mu_2}^{\mu_1} \cdots \sum_{\eta_h = \mu_{h+1}}^{\mu_h} f(\eta_1, \dots, \eta_h) = p \cdot s(\mu f). \end{aligned}$$ ∎

Remark: Let $a(x_1, \dots, x_h)$ be p.h.d. of degree d; then

$$p \cdot i(\mu;\ f) = p(\mu_1, \dots, \mu_{h+1})$$

is p.h.d. of degree $d + h$. Assume now that $\mu \vdash n$ and denote

$$\mu_i = \frac{n}{h+1} + c_i\sqrt{n}, \quad c_i = c_i(\mu), \quad i = 1, \dots, h+1.$$

Then $(\mu_i \to \mu_i - \frac{n}{h+1})$

$$p(\mu_1, \dots, \mu_{h+1}) = p(c_1\sqrt{n}, \dots, c_{h+1}\sqrt{n}) = \sqrt{n}^{d+h} p(c_1, \dots, c_{h+1}).$$

Moreover, if $\mu_1 > \cdots > \mu_{h+1}$ then $c_1 > \cdots > c_{h+1}$, hence $p(c_1, \dots, c_{h+1}) > 0$.

In Proposition 4.10 we have summarized the above discussion; it implies that p.h.d. polynomials are "nice", and this implies our Theorem 4.3.

PROPOSITION 4.10: *Let $a = a(x_1, \dots, x_k)$ be a p.h.d. polynomial of degree d, let $\mu = (\mu_1, \dots, \mu_{k+1})$ be a partition, and denote*

$$b(\mu) = p \cdot s(\mu;\ a) = \sum_{\lambda_1 = \mu_2}^{\mu_1} \cdots \sum_{\lambda_k = \mu_{k+1}}^{\mu_k} a(\lambda_1, \dots, \lambda_k),$$

$$p(\mu) = p \cdot i(\mu;\ a) = \int_{\mu_2}^{\mu_1} dx_1 \cdots \int_{\mu_{k+1}}^{\mu_k} dx_k \cdot a(x_1, \ldots, x_k).$$

Write $b(\mu) = p(\mu) + \bar{r}(\mu)$. Then all three terms are polynomials of the differences $\mu_i - \mu_j$, $b(\mu)$ and $p(\mu)$ are of degree $d+k$ while the degree of $\bar{r}(\mu)$ is at most $d+k-1$. We have $p(\mu) = \sqrt{n}^{d+k} p(c)(c = c(\mu))$, and if $\mu_1 > \cdots > \mu_{k+1}$ then $p(c) > 0$. Also, $p(z)$ is p.h.d.

NOTE: Expanding $\bar{r}(\mu)$ as a sum of its homogeneous terms, we similarly have

$$\bar{r}(\mu) = \sum_{j=0}^{d+r-1} \sqrt{n}^j \bar{r}_j(c_1, \ldots, c_{k+1}).$$

It clearly follows now that, in the notation of [8, 3.5], such an $a(x)$ is "nice", hence Theorem 3.7 of [8] applies to yield Theorem 4.3 of the present paper.

Alternatively, it clearly follows from the above that

$$\sum_{\mu \in \Lambda_{k+1}(n)} b(\mu) d_\mu \underset{n \to \infty}{\simeq} \sum_{\mu \in \Lambda_{k+1}(n)} p(\mu) d_\mu.$$

The asymptotics of both sides are done in [8] (follow the proof of Theorem 3.7 there), which leads to Theorem 4.3 here.

Before proving Theorem 4.5, note that 4.6–4.10 clearly imply

COROLLARY 4.11: *Let $a(x_1, \ldots, x_h)$, $b(x_1, \ldots, x_h)$ be two polynomials of the $x_i - x_j$'s such that $a(x)$ is p.h.d. and $b(x) = a(x)+$ lower terms in the $x_i - x_j$'s. Then $p \cdot s(z; b)$ is a polynomial of the $z_i - z_j$'s, and $p \cdot s(\mu; b) = p \cdot i(\mu; a)+$ lower terms in the $\mu_i - \mu_j$'s (i.e. $p \cdot s(\mu, b) \approx p \cdot i(\mu; a)$).*

The proof of Theorem 4.5: By [2], Theorem 1.2,

$$b^{(q-1)}(\lambda) = F(\lambda_q, \lambda_{2q}, \ldots, \lambda_{kq}) \cdot \left(\prod_{t=0}^{k} \prod_{tq+1 \le i < j \le tq+q-1} (\lambda_i - \lambda_j + j - i) \right) \cdot (1!2! \cdots (q-2)!)^{-k-1},$$

hence $b^{(q-1)}(\lambda)$ is a polynomial of the $\lambda_i - \lambda_j$'s and $b^{(q-1)}(\lambda) = a^{(0)}(\lambda)+$ lower terms in the $\lambda_i - \lambda_j$'s. Also, $a^{(0)}(x)$ is p.h.d., hence 4.5 holds for $\ell = q-1$, while 4.6 implies (4.5.2) for all s.

Proceed by induction on $q - 1 \le \ell$.

By definition of "p.s" and by Theorem 1.3 of [8], $b^{(\ell+1)}(\mu) = p \cdot s(\mu; b^{(\ell)})$, hence (4.5.1) follows by induction from 4.9.

Finally, (4.5.3) easily follows by induction from Corollary 4.11:

$$b^{(\ell+1)}(\mu) = p \cdot s(\mu; b^{(\ell)}) = p \cdot s(\mu; a^{(\ell-q+1)} + \text{ lower terms}),$$

$$p \cdot i(\mu; a^{(\ell-q+1)}) + \text{ lower terms in } \mu_i - \mu_j \text{ 's}$$
$$= a^{(\ell-q+2)}(\mu) + \text{ lower terms in } \mu_i - \mu_j \text{ 's.} \quad \blacksquare$$

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